

ON THE QUOTIENT OF \mathbb{C}^4 BY A FINITE PRIMITIVE GROUP OF TYPE (I)

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ABSTRACT. We study rationality problem for the quotient of \mathbb{C}^4 by a finite primitive group G of Type (I). We prove that this quotient is a rational variety for any such G .

1. INTRODUCTION

Given a complex affine space $\mathbb{C}^n = \text{Spec}(\mathbb{C}[x_1, \dots, x_n])$ and a finite group G acting linearly on \mathbb{C}^n , one of the fundamental questions to ask is whether the field of G -invariant rational functions on \mathbb{C}^n is also a purely transcendental extension of \mathbb{C} , or, in other words, whether the variety \mathbb{C}^n/G is rational (see [3] (and references therein) for an extensive overview of the current state of the problem). By a simple argument (see [3, Proposition 1.2]), one can show that \mathbb{C}^n/G is birationally isomorphic to $(\mathbb{P}(\mathbb{C}^n)/G) \times \mathbb{P}^1$, so $n = 4$ is the first non-trivial issue, since the Lüroth problem has a positive solution for $n \leq 3$. The case of $n = 4$ has been treated in detail in [3]. However, for some of the groups G the (non-)rationality of \mathbb{C}^4/G was not established.

Namely, let $\mathbb{O}, \mathbb{I} \subset SL_2(\mathbb{C})$ be the octahedron and icosahedron subgroups, respectively. Identify $U_0 := \mathbb{C}^4$ with the space of (2×2) -matrices $A := \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$, $X_i \in \mathbb{C}$, and consider the action of the group $G := \mathbb{O} \times \mathbb{I}$ on U_0 such that \mathbb{O} and \mathbb{I} act by multiplying A from the left and right, respectively. Furthermore, by the above argument, in order to establish rationality of U_0/G one may assume that $G := (\mathbb{O} \times \mathbb{I}) \cdot \mathbb{C}^*$ for the standard diagonal action of \mathbb{C}^* on U_0 . Then for this group action we prove the following

Theorem 1.1. *The 3-fold U_0/G is rational.*

Theorem 1.1 settles the remaining case in [3] of quotients of \mathbb{P}^3 (or, equivalently, \mathbb{C}^4) by *finite primitive groups of Type (I)* (see [3, Section 2] for the description of these).

Let us outline the proof of Theorem 1.1. Recall that in [3], after taking the \mathbb{C}^* -quotient of U_0 and passing to the projectivized G -action on \mathbb{P}^3 , with G now equal $\mathbb{O} \times \mathbb{I}$, one can notice that \mathbb{P}^3/G is birationally isomorphic to $SL_2(\mathbb{C})/G$ for the induced G -action on $SL_2(\mathbb{C}) \subset U_0$. Further, compactifying $SL_2(\mathbb{C})$ by a smooth Fano 3-fold W with either \mathbb{O} - or \mathbb{I} -action, one might try to prove that the corresponding quotient of W is rational by finding an equivariant birational map of W onto a product of positive-dimensional varieties (see [3, Section 2], where this idea worked perfectly well for all finite primitive groups of Type (I), except for the given G).

Our approach is more direct (and simpler in a sense). Namely, a natural generalization of the construction of \mathbb{P}^1 leads to a smooth projective compactification V of U_0 (see **2.1**, **2.2** and **2.7** below). Moreover, V turns out to be a Fano 4-fold with Picard number 1 and Fano index 4, i.e., V is a quadric in \mathbb{P}^5 by a result due to T. Fujita (see **2.9**). Furthermore, the G -action on U_0 extends to a regular action on V (see **2.12**), and it is easy to see that there are two G -invariant hyperplane sections of $V \subset \mathbb{P}^5$. This makes one hope, by considering the corresponding G -equivariant linear projection $V \dashrightarrow \mathbb{P}^1$, to split the threefold V/G into a product of positive-dimensional varieties, thus proving rationality of V/G (and U_0/G as well). Though we did not succeed on this way, a slight modification of the construction of V leads to a quadratic cone V' in \mathbb{P}^5 which compactifies $U_0/(\mathbb{Z}/2\mathbb{Z})^{(1)}$ (the group $\mathbb{Z}/2\mathbb{Z}$ acts on U_0 via multiplying every X_i by -1) and such that the G -action on $U_0/(\mathbb{Z}/2\mathbb{Z})$ extends to a regular action on V' (see **3.1**). Moreover, $V' \subset \mathbb{P}^5$ happens to have three linearly independent G -invariant hyperplane sections (see Lemma 3.3), and the development of the preceding idea with projection enables one to prove that the threefold V'/G is rational (see Lemma 3.4). It is easy to deduce from this that the threefold V/G is also rational (see Lemma 3.5).

¹⁾By “ V' compactifies $U_0/(\mathbb{Z}/2\mathbb{Z})$ ” we mean that $\mathbb{C}(V') = \mathbb{C}(U_0/(\mathbb{Z}/2\mathbb{Z}))$ for the fields of meromorphic functions.

Remark 1.2. Instead of $\mathbb{O} \times \mathbb{I}$ one may take any other finite primitive group G of Type (I) and prove that the corresponding quotient \mathbb{C}^4/G is rational, repeating literally the arguments in Sections 2 and 3 below. This gives another proof of Theorem 2.1 in [3].

Notation. We use standard notions and facts from [2]. All varieties, if not specified, are assumed to be algebraic. Also throughout the paper we use the following notation:

- (†) Given two varieties X and Y , $X \approx Y$ denotes birational equivalence between them. For an algebraic group G acting regularly on both X and Y , we write $X \approx_G Y$ if there exists a G -equivariant birational map $X \dashrightarrow Y$.

2. ONE EXPLICIT COMPACTIFICATION

2.1. Gluing construction. Take another copy U_1 of \mathbb{C}^4 . Identify U_1 with the space of (2×2) -matrices, as U_0 above. Let $\varphi_1 : U_0 \dashrightarrow U_1$ be birational map induced by the morphism $GL_2(\mathbb{C}) \rightarrow GL_2(\mathbb{C})$ which sends every invertible matrix $A \in U_0$ to $A^{-1} \in U_1$. Set $X_i^{(1)} := \varphi_1^{-1*}(X_i)$, $1 \leq i \leq 4$. These extend to affine coordinates on U_1 . Put also $\Delta_0 := \det A$ and $\Delta_1 := \varphi_1^{-1*}(\Delta_0)$.

Further, let $l_{\alpha,\beta}$ be the linear automorphism of U_0 which permutes X_α and X_β in A with $\alpha + \beta \neq 5$. Take another copy $U_{\alpha,\beta}$ of \mathbb{C}^4 , as U_0 and U_1 above, and consider birational map $\varphi_{\alpha,\beta} := \varphi_1 \circ l_{\alpha,\beta} : U_0 \dashrightarrow U_{\alpha,\beta}$. Set $X_i^{(\alpha,\beta)} := \varphi_{\alpha,\beta}^{-1*}(X_i)$. These extend to affine coordinates on $U_{\alpha,\beta}$. Put also $\Delta_{\alpha,\beta} := \varphi_{\alpha,\beta}^{-1*}(\Delta_0)$.

Now glue $U_0, U_1, U_{\alpha,\beta}$ together via the maps $\varphi_1, \varphi_{\alpha,\beta}$ for various α, β . We get a smooth complex 4-fold V so that $U_0, U_1, U_{\alpha,\beta}$ are analytic domains covering V . Note that $\Delta_1 = \Delta_0^{-1}$ on $U_0 \cap U_1$ and $\Delta_{\alpha,\beta} = l_{\alpha,\beta}^*(\Delta_0)$ on $U_0 \cap U_{\alpha,\beta}$.

2.2. Compactness of V . Consider a sequence of points on $U_0 \subset V$ of the form

$$O_t := \begin{pmatrix} \alpha_1 t^{d_1} & \alpha_2 t^{d_2} \\ \alpha_3 t^{d_3} & \alpha_4 t^{d_4} \end{pmatrix},$$

$t \in \mathbb{C}$, for some fixed $\alpha_i \in \mathbb{C}, d_i \in \mathbb{Z}$. By the valuative criterion of properness, V is compact provided that every such sequence is convergent on V . Without loss of generality, in what follows we assume that $t \rightarrow \infty$.

Lemma 2.3. *If $\alpha_i = \alpha_j = 0$ for some $i \neq j$, then the sequence $\{O_t\}_{t \in \mathbb{C}}$ is convergent on V .*

Proof. Assume first that $\alpha_3 = \alpha_4 = 0$. Then all points on V of the form $\begin{pmatrix} \lambda & \mu \\ 0 & 0 \end{pmatrix}$, $\lambda, \mu \in \mathbb{C}$, glue into a surface $S \simeq \mathbb{P}^1 \times \mathbb{P}^1 \subset V$. Indeed, we have

$$\begin{pmatrix} \lambda & \mu \\ 0 & 0 \end{pmatrix} = \varphi_{\alpha,\beta}^{-1} \left(\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \mu^{-1} \end{pmatrix} \right)$$

for $\lambda, \mu \in \mathbb{C}^*$ and appropriate α, β , which implies that the point $\begin{pmatrix} \lambda & \mu \\ 0 & 0 \end{pmatrix}$ is identified with $\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \mu^{-1} \end{pmatrix}$ when gluing U_0 and $U_{\alpha,\beta}$ via $\varphi_{\alpha,\beta}$. In particular, $\{O_t\}_{t \in \mathbb{C}}$ is convergent on V , since $O_t \in S$ for all t . Finally, one treats the cases of other i, j in exactly the same way, by considering the gluings of U_0 with other charts $U_{\alpha,\beta}$. \square

Lemma 2.4. *If $d_i > \max_{j \neq i} \{d_j\}$ and $\alpha_i \neq 0$, then the sequence $\{O_t\}_{t \in \mathbb{C}}$ is convergent on V .*

Proof. We may assume that $i = 1$ and $d_1 > 0$. Then if all $d_j < 0$ (for $\alpha_j \neq 0$), it follows from Lemma 2.3 that the limit $\lim_{t \rightarrow \infty} O_t$ equals $\begin{pmatrix} \infty & 0 \\ 0 & 0 \end{pmatrix} \in V$. Hence we may assume that $d_{j'} \geq 0$ (and $\alpha_{j'} \neq 0$) for some $j' > 1$. Further, by passing to an appropriate chart $U_{\alpha,\beta}$ we may additionally assume that $j' = 4$. Then every point O_t is identified with

$$\Delta^{-1} \begin{pmatrix} \alpha_4 t^{d_4} & -\alpha_3 t^{d_3} \\ -\alpha_2 t^{d_2} & \alpha_1 t^{d_1} \end{pmatrix},$$

where $\Delta := \alpha_1 \alpha_4 t^{d_1+d_4} - \alpha_2 \alpha_3 t^{d_2+d_3} \gg t^{d_1}$ as $t \rightarrow \infty$, from which it is straightforward that $\{O_t\}_{t \in \mathbb{C}}$ is convergent on V . \square

Lemma 2.4 implies that one can put $d_i = 1$ for all i . Furthermore, similar argument as in the proof of Lemma 2.4 shows that one can also put $\alpha_i = 1$ for all i . Under these assumptions we get

Lemma 2.5. *The sequence $\{O_t\}_{t \in \mathbb{C}}$ is convergent on V .*

Proof. For every t , we have

$$(2.6) \quad O_t = \begin{pmatrix} t & t \\ t & t \end{pmatrix} = \lim_{\varepsilon \rightarrow 1} \begin{pmatrix} \varepsilon t & t \\ t & t \end{pmatrix}$$

on V . On the other hand, for ε fixed, the limit $\lim_{t \rightarrow \infty} \begin{pmatrix} \varepsilon t & t \\ t & t \end{pmatrix}$ equals $\begin{pmatrix} \infty & \infty \\ \infty & \infty \end{pmatrix} \in V$ by the preceding arguments. Then we can set

$$\lim_{t \rightarrow \infty} O_t := \lim_{\varepsilon \rightarrow 1} \lim_{t \rightarrow \infty} \begin{pmatrix} \varepsilon t & t \\ t & t \end{pmatrix} = \begin{pmatrix} \infty & \infty \\ \infty & \infty \end{pmatrix}$$

on V . It is easy to see that the limit of $\{O_t\}_{t \in \mathbb{C}}$ does not depend on the approximating sequence (2.6). \square

Thus, V is compact.

2.7. Projectivity of V . It is enough to find a positive line bundle on V . Let D be a prime Cartier divisor on V with the equations $\Delta_0 = 0$ on U_0 and $\Delta_{\alpha,\beta} = 0$ on $U_{\alpha,\beta}$ for all α, β (cf. **2.1**). Set $\mathcal{L} := \mathcal{O}_V(D)$. The line bundle \mathcal{L} carries a hermitian metric $|\cdot|$ such that $1 = |\Delta_0| = |\Delta_{\alpha,\beta}|$ on $U_0 \cap U_1$ and $U_0 \cap U_{\alpha,\beta}$ for all α, β .

Lemma 2.8. *\mathcal{L} is a positive line bundle.*

Proof. Let $\theta \in H^0(V, \mathcal{L})$ be the global section such that $(\theta)_0 = D$. Put $\theta_0 := \theta|_{U_0}$, $\theta_1 := \theta|_{U_1}$, $\theta_{\alpha,\beta} := \theta|_{U_{\alpha,\beta}}$.

Restrict \mathcal{L} to U_0 and define a metric h_0 on $\mathcal{L}|_{U_0}$ as follows:

$$h_0 := (1 + |X_1|^2)|\theta_0|.$$

On $U_0 \cap U_1$ we have

$$|\theta_1| = |\theta_0| \frac{1}{|\Delta_0|} = |\theta_0|,$$

and hence

$$h_0 = |\theta_1| + \frac{|X_1|^2}{|\Delta_0|^2} |\theta_1| = (1 + |X_1^{(1)}|^2) |\theta_1|.$$

This extends h_0 to a metric on \mathcal{L} over $U_0 \cup U_1$. Repeating the same construction, with U_1 replaced by $U_{\alpha,\beta}$, we obtain a global metric on \mathcal{L} , equal to

$$(1 + |X_1^{(\alpha,\beta)}|^2) |\theta_{\alpha,\beta}|$$

on each $U_{\alpha,\beta}$. Moreover, starting with the metric

$$h := |\theta_0| \prod_{i=1}^4 (1 + |X_i|^2)^{1/4}$$

on \mathcal{L} over U_0 , the same argument yields to a metric²⁾ on \mathcal{L} which extends h . Let us denote this global metric again by h .

Now, in order to prove that \mathcal{L} is positive it is enough to show that $-\sqrt{-1}\Theta > 0$ for the $(1,1)$ -form $\Theta := \frac{\sqrt{-1}}{2\pi} \bar{\partial} \partial \log h \in c_1(\mathcal{L})$. Further, in verifying that $-\sqrt{-1}\Theta > 0$ we restrict ourselves to the chart U_0 , since for U_1 and $U_{\alpha,\beta}$ the arguments are exactly the same. Finally, on U_0 we have

$$-\sqrt{-1}\Theta = \frac{1}{8\pi} \sum_{i=1}^4 \frac{dX_i \wedge d\bar{X}_i}{(1 + |X_i|^2)^2} > 0,$$

hence the result. \square

²⁾Equal to $|\theta_{\alpha,\beta}| \prod_{i=1}^4 (1 + |X_i^{(\alpha,\beta)}|^2)^{1/4}$ on each $U_{\alpha,\beta}$.

2.9. V is a quadric 4-fold. There is a unique (prime) Cartier divisor $D_\infty \sim D$ on V with the equation $\Delta_1 = 0$ on U_1 . Indeed, one may define D_∞ by taking the closure in V of the locus $(\Delta_1 = 0) \subset U_1$, and $D_\infty \sim D$ via the rational map $V \dashrightarrow \mathbb{P}^1$ which extends the map $A \mapsto \det A$ on U_0 . Equivalently, one may notice that the divisors D_∞ and $D + (f)$ determine the same discrete valuations of the function field $\mathbb{C}(V)$, where f is the rational function on V equal to Δ_0^{-1} on U_0 (cf. Remark 2.11 below). Note that $D \neq D_\infty$ (cf. the similar construction of \mathbb{P}^1 and of the divisors $0, \infty \in \mathbb{P}^1$).

Lemma 2.10. $K_V \sim -4D$.

Proof. Let us start with the form $\omega := dX_1 \wedge dX_2 \wedge dX_3 \wedge dX_4$ on U_0 . We have

$$\hat{X}_j := d\left(\frac{X_j}{X_1X_4 - X_2X_3}\right) = \frac{dX_j}{X_1X_4 - X_2X_3} - \frac{X_j d(X_1X_4 - X_2X_3)}{(X_1X_4 - X_2X_3)^2}$$

for all j . It is easy to see that

$$\begin{aligned} \hat{X}_1 \wedge \hat{X}_2 \wedge \hat{X}_3 \wedge \hat{X}_4 &= \frac{dX_1 \wedge dX_2 \wedge dX_3 \wedge dX_4}{(X_1X_4 - X_2X_3)^4} - \frac{\sum_{1 \leq j \leq 4} X_j d(X_1X_4 - X_2X_3) dX_1 \wedge \dots \wedge d\hat{X}_j \wedge \dots \wedge dX_4}{(X_1X_4 - X_2X_3)^5} = \\ &= \frac{dX_1 \wedge dX_2 \wedge dX_3 \wedge dX_4}{(X_1X_4 - X_2X_3)^4}. \end{aligned}$$

Then we get

$$dX_1 \wedge dX_2 \wedge dX_3 \wedge dX_4 = \frac{1}{\Delta_1^4} dX_1^{(1)} \wedge dX_2^{(1)} \wedge dX_3^{(1)} \wedge dX_4^{(1)}$$

on $U_0 \cap U_1$. This extends ω to a meromorphic form on $U_0 \cup U_1$. Note that $K_V = -4D_\infty \sim -4D$ on $U_0 \cup U_1$.

Repeating the same construction, with U_1 replaced by $U_{\alpha,\beta}$, we obtain a global meromorphic section of the line bundle $\mathcal{O}_V(K_V)$, equal to

$$\frac{1}{l_{\alpha,\beta}^*(\Delta_{\alpha,\beta})^4} dX_1^{(\alpha,\beta)} \wedge dX_2^{(\alpha,\beta)} \wedge dX_3^{(\alpha,\beta)} \wedge dX_4^{(\alpha,\beta)}$$

on $U_0 \cap U_{\alpha,\beta}$ for each α, β . Hence $K_V = -4D_\infty \sim -4D$ on V . \square

Lemmas 2.8, 2.10 and [2, Theorem 3.1.14] imply that V is a quadric 4-fold in \mathbb{P}^5 (with D, D_∞ being irreducible hyperplane sections of $V \subset \mathbb{P}^5$).

Remark 2.11. It follows from the proof of Lemma 2.10 that the equation of the divisor D_∞ on $U_{\alpha,\beta}$ is $l_{\alpha,\beta}^*(\Delta_{\alpha,\beta}) = 0$ for each α, β .

2.12. G -action on V . Let us show that the G -action on U_0 extends to a regular action on V . By construction of V , every $g \in G$ determines a birational automorphism $g : V \dashrightarrow V$, regular and bijective on $U_0 \cup U_1$. Furthermore, we have $V \setminus (U_0 \cup U_1) \subseteq D_0 \cup D_\infty$, since

$$U_0 \cup U_1 \supseteq V \setminus (D_0 \cup D_\infty) = U_0 \cap U_1 \cap \bigcap_{\alpha,\beta} U_{\alpha,\beta}$$

(see 2.1 and the equations of D_0, D_∞). Then, since $g(D \cap U_0) = D \cap U_0$, $g(D_\infty \cap U_1) = D_\infty \cap U_1$ and D_0, D_∞ are irreducible, we obtain that g is an isomorphism in codimension 2 on V , hence $g_*(D) = D$ and $g_*(D_\infty) = D_\infty$ in $\text{Pic}(V)$. This implies that g is induced from a regular automorphism of $\mathbb{P}^5 \supset V$. Hence we get $g \in \text{Aut}(V)$ and $U_0/G \approx V/G$ (note that $\mathbb{C}(U_0) = \mathbb{C}(V)$ by construction of V).

Remark 2.13. The fact that $V \supset U_0$ is the quadric with a G -action satisfying the above properties is the main reason for running preceding constructions. At the same time, as was communicated to us by Yu. G. Prokhorov, there is a much more easy way to get V and G as above. Indeed, take the standard compactification of U_0 by \mathbb{P}^4 , with the divisor $B \subset \mathbb{P}^4$ at infinity, and extend the G -action to \mathbb{P}^4 in the usual way. Then there is a G -invariant smooth quadric $S \subset B = \mathbb{P}^3$. Let $\sigma : Y \rightarrow \mathbb{P}^4$ be the blow up of S with the exceptional divisor $E := \sigma^{-1}(S)$. It is easy to see that the linear system $|2L - E|$, where $L := \sigma^*(B)$, determines a birational contraction $\tilde{\sigma} : Y \rightarrow \tilde{Y}$, mapping the proper transform $\sigma_*^{-1}(B) \sim L - E$ of the divisor B to a point. Moreover, since the normal bundle of $\sigma_*^{-1}(B) \simeq \mathbb{P}^3$ on Y is $\mathcal{O}_{\mathbb{P}^3}(-1)$, one easily gets that $\tilde{\sigma}$ is the blow up of a smooth point on \tilde{Y} . Furthermore, \tilde{Y} is a (smooth) Fano 4-fold with $\text{Pic}(\tilde{Y}) = \mathbb{Z} \cdot \tilde{\sigma}_*(L)$ and such that $\tilde{\sigma}^*(K_{\tilde{Y}}) = K_Y - 3\sigma_*^{-1}(B) = -4L$, i.e., the Fano index of \tilde{Y} is 4. Hence, as V above, \tilde{Y} is a smooth quadric in \mathbb{P}^5 . Finally, the construction of \tilde{Y} implies that both σ and

$\tilde{\sigma}$ are G -equivariant. Hence \tilde{Y} is a G -equivariant compactification of U_0 as above. However, we keep our original construction, since we could not provide a similar explicit birational construction, as for V above, for the 4-fold V' below.

3. PROOF OF THEOREM 1.1

3.1. Gluing construction revisited. In the notation of Section 2, let us replace each U_i and $U_{\alpha,\beta}$ with $U_i/(\mathbb{Z}/2\mathbb{Z})$ and $U_{\alpha,\beta}/(\mathbb{Z}/2\mathbb{Z})$, respectively, where $\mathbb{Z}/2\mathbb{Z}$ acts via $X_i \mapsto -X_i$, $1 \leq i \leq 4$. Note that the gluing maps φ_1 and $\varphi_{\alpha,\beta}$ are $(\mathbb{Z}/2\mathbb{Z})$ -equivariant. Hence we can glue the given six copies of $\mathbb{C}^4/(\mathbb{Z}/2\mathbb{Z})$ together via $\varphi_1, \varphi_{\alpha,\beta}$ for various α, β as above. We get an algebraic space V' (with $\{U_0, U_1, U_{\alpha,\beta}\}_{\alpha,\beta}$ being an open cover of V' in the orbifold topology), which, similar to **2.2**, is a compact complex 4-fold. Note that V' has (only) isolated singularities. One can also see (directly from the definition) that the singularities of V' are all terminal.

Remark 3.2. Note that the gluing maps $\varphi_1, \varphi_{1,2}, \dots$ are rather *algebraic* (see [1]) than analytic. Indeed, $\varphi_1, \varphi_{1,2}, \dots$, when lifted to universal covers ($= \mathbb{C}^4$) of the charts $U_0 = \mathbb{C}^4/(\mathbb{Z}/2\mathbb{Z}), \dots$, are only $\mathbb{Z}/2\mathbb{Z}$ -equivariant, but not $\mathbb{Z}/2\mathbb{Z}$ -invariant (for the latter, the defining functions of lifted $\varphi_1, \varphi_{1,2}, \dots$ being non- $\mathbb{Z}/2\mathbb{Z}$ -invariant, see **2.1**). It is easy to see, however, that the complex structure on V' is provided by the charts $U_0 \cup U_1, U_0 \cup U_{1,2}, \dots$ (but not by $\{U_0, U_1, U_{\alpha,\beta}\}_{\alpha,\beta}$), glued via $\varphi_1, \varphi_{1,2}, \dots$. Note also that $\mathbb{C}(V') = \mathbb{C}(U_0)$ (at the same time, as one can easily see, the map $\mathbb{C}^4 \rightarrow U_0$ does not induce a *regular* map $V \rightarrow V'$, for above V , so that V' is only *birationally* a quotient $V/(\mathbb{Z}/2\mathbb{Z})$).

Further, since the defining equations of the divisor D in **2.7** are $(\mathbb{Z}/2\mathbb{Z})$ -invariant, we get a prime Cartier divisor D' on V' . It follows from the arguments in **2.7** that the line bundle $\mathcal{O}_{V'}(D')$ possesses a (orbifold) hermitian metric with positive curvature. Hence D' is ample. Furthermore, since the defining equations of the divisor D_∞ are $(\mathbb{Z}/2\mathbb{Z})$ -invariant, we get a prime Cartier divisor $D'' \neq D'$ on V' with $D'' \sim D'$ (see **2.9** and Remark 2.11). Then the construction of K_V in the proof of Lemma 2.10 implies that V' is a Fano 4-fold with $K_{V'} = -4D'' \sim -4D'$. Thus, again by [2, Theorem 3.1.14], V' is a quadratic cone in \mathbb{P}^5 with a vertex O . Furthermore, similar to **2.12**, the G -action on $U_0 = \mathbb{C}^4/(\mathbb{Z}/2\mathbb{Z})$ extends to a regular action on V' so that $\mathbb{C}^4/(G \times \mathbb{Z}/2\mathbb{Z}) \approx V'/G$.

Lemma 3.3. *The space $H^0(V', \mathcal{O}_{V'}(D'))$ contains three linearly independent G -invariant elements.*

Proof. The construction of D and D_∞ implies that D' and D'' are G -invariant (cf. **2.12**). Moreover, since D' and D'' are hyperplane sections of $V' \subset \mathbb{P}^5$ passing through O ,³⁾ there is also a smooth G -invariant hyperplane section of V' . This gives three linearly independent G -invariant elements in $H^0(V', \mathcal{O}_{V'}(D'))$. \square

Lemma 3.4. *The 3-fold V'/G is rational.*

Proof. By Lemma 3.3, we may assume the equation of $V' \subset \mathbb{P}^5 = \text{Proj}(\mathbb{C}[x_0, \dots, x_5])$ to be $x_0x_1 + x_2x_3 + x_4^2 = 0$, with $\mathbb{C}^* \subset G$ acting diagonally and $\mathbb{O} \times \mathbb{I} \subset G$ fixing x_0, x_1, x_5 . Let $V' \dashrightarrow \mathbb{P}^2$ be restriction to V' of the linear projection from the G -invariant plane $\Pi := (x_2 = x_3 = x_4 = 0)$. Note that $V' \cap \Pi$ is a pair of distinct lines (with trivial $\mathbb{O} \times \mathbb{I}$ -action). Then, blowing up V' at $V' \cap \Pi$, we get a normal 4-fold $V'' \approx_G V'$ together with a G -equivariant morphism $V'' \rightarrow \mathbb{P}^2$, which has (at least 3) G -invariant sections and generic fiber \approx [a quadratic cone]. In particular, we get

$$V' \approx_G [\text{quadratic cone with trivial } (\mathbb{O} \times \mathbb{I})\text{-action}] \times \mathbb{P}^2,$$

which implies that V'/G is rational. \square

Lemma 3.5. *The 3-fold V/G is rational.*

Proof. We have

$$\mathbb{C}^4/G = \mathbb{C}^4/(\mathbb{O} \times \mathbb{I} \times \mathbb{C}^*) \simeq \mathbb{C}^4/(\mathbb{O} \times \mathbb{I} \times \mathbb{C}^* \times \mathbb{Z}/2\mathbb{Z}) = U_0/G$$

for the (non-canonical) isomorphism $\mathbb{C}^* \simeq \mathbb{C}^*/(\mathbb{Z}/2\mathbb{Z})$. Now the statement follows from Lemma 3.4 because $\mathbb{C}(V'/G) = \mathbb{C}(U_0/G)$ by the above construction of V' . \square

³⁾Indeed, we have $D \cap U_0 = (X_1X_4 - X_2X_3 = 0)$, hence $O \in D'$, and similarly for D'' on U_1 .

Lemma 3.5 proves Theorem 1.1.

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